Greedy Algorithms: Introduction

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Outline

1. Maximize Your Salary
2. Queue of Patients
3. Implementation and Analysis
4. Main Ingredients

What's Coming

- Solve salary maximization problem
- Come up with a greedy algorithm yourself
- Solve optimal queue arrangement problem
- Generalize solutions using the concepts of greedy choice, subproblem and safe choice

Maximize Salary

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Largest Number

Toy problem

What is the largest number that consists of digits 9, 8, 9, 6, 1? Use all the digits.

Largest Number

Toy problem

What is the largest number that consists of digits 9, 8, 9, 6, 1? Use all the digits.

Examples

$$16899, 69891, 98961, \ldots$$

Correct answer

99861

Greedy Strategy

$$\{9, 8, 9, 6, 1\} \longrightarrow \text{?????}$$

Greedy Strategy

Find max

$\{9, 8, 9, 6, 1\} \longrightarrow$

- Find max digit

Greedy Strategy

Find max

$\{9, 8, 9, 6, 1\} \longrightarrow$

- Find max digit

Greedy Strategy

Find max | Append

$\{9, 8, 9, 6, 1\} \longrightarrow$

- Find max digit
- Append it to the number

Greedy Strategy

Find max　　　　　　Append

$\{9, 8, 9, 6, 1\} \longrightarrow 9$

- Find max digit
- Append it to the number

Greedy Strategy

Find max		Append
$\{9, 8, 9, 6, 1\}$	$\longrightarrow$	9
Remove	

- Find max digit
- Append it to the number
- Remove it from the list of digits

Greedy Strategy

Find max		Append
$\{9, 8, 9, 6, 1\}$	$\longrightarrow$	9
Remove	

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Greedy Strategy

Find max		Append
$\{8, 9, 6, 1\}$	$\longrightarrow$	9
Remove	

- Find max digit
- Append it to the number
- Remove it from the list of digits
- Repeat while there are digits in the list

Greedy Strategy

Find max   Append

$\{8, 9, 6, 1\} \longrightarrow 9$

Remove

- Find max digit
- Append it to the number
- Remove it from the list of digits
- Repeat while there are digits in the list

Greedy Strategy

Find max		Append
$\{8, 9, 6, 1\}$	$\longrightarrow$	99
Remove	

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Greedy Strategy

Find max		Append
$\{8, 9, 6, 1\}$	$\longrightarrow$	99
Remove	

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Greedy Strategy

Find max	Append
$\{8, 6, 1\} \longrightarrow 99$
Remove

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Greedy Strategy

Find max		Append
$\{8, 6, 1\}$	$\longrightarrow$	99
Remove		

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Greedy Strategy

Find max		**Append**
$\{8, 6, 1\}$	$\longrightarrow$	998
Remove	

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Greedy Strategy

Find max

$\{8, 6, 1\} \longrightarrow 998$

Append

Remove

- Find max digit
- Append it to the number
- Remove it from the list of digits
- Repeat while there are digits in the list

Greedy Strategy

Find max		Append
$\{6, 1\}$	$\longrightarrow$	998
Remove	

- Find max digit
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- Repeat while there are digits in the list

Greedy Strategy

Find max		Append

Find max		Append
$\{6, 1\}$	$\longrightarrow$	998
Remove		

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Greedy Strategy

Find max		Append
$\{6, 1\}$	$\longrightarrow$	9986

Remove

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- Repeat while there are digits in the list

Greedy Strategy

Find max

Append

$\{6, 1\} \longrightarrow 9986$

Remove

- Find max digit
- Append it to the number
- Remove it from the list of digits
- Repeat while there are digits in the list

Greedy Strategy

Find max — Append

$\{1\} \longrightarrow$ 9986

Remove

- Find max digit
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Greedy Strategy

Find max Append

$\{1\} \longrightarrow 9986$

Remove

- Find max digit
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Greedy Strategy

Find max

Append

$\{1\} \longrightarrow 99861$

Remove

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Greedy Strategy

Find max

Append

$\{1\} \longrightarrow 99861$

Remove

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- Append it to the number
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- Repeat while there are digits in the list

Greedy Strategy

Find max

Append

$\{\} \longrightarrow 99861$

Remove

- Find max digit
- Append it to the number
- Remove it from the list of digits
- Repeat while there are digits in the list

Greedy Strategy

Find max		Append
$\{9, 8, 9, 6, 1\}$	$\longrightarrow$	99861
Remove		Success!

- Find max digit
- Append it to the number
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- Repeat while there are digits in the list

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Queue of Patients

Queue Arrangement

Input: n patients have come to the doctor's office at 9:00AM. They can be treated in any order. For i-th patient, the time needed for treatment is t_i. You need to arrange the patients in such a queue that the total waiting time is minimized.

Output: The minimum total waiting time.

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(1, 2, 3)$:

- First patient doesn't wait

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(1, 2, 3)$:

- First patient doesn't wait
- Second patient waits for 15 minutes

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(1, 2, 3)$:

- First patient doesn't wait
- Second patient waits for 15 minutes
- Third patient waits for $15 + 20 = 35$ minutes

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(1, 2, 3)$:

- First patient doesn't wait
- Second patient waits for 15 minutes
- Third patient waits for $15 + 20 = 35$ minutes
- Total waiting time $15 + 35 = 50$ minutes

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(3, 1, 2)$:

- First patient doesn't wait

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.

Arrangement $(3, 1, 2)$:

- First patient doesn't wait
- Second patient waits for 10 minutes

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.
Arrangement $(3, 1, 2)$:

- First patient doesn't wait
- Second patient waits for 10 minutes
- Third patient waits for $10 + 15 = 25$ minutes

Optimal Queue Arrangement

$t_1 = 15$, $t_2 = 20$ and $t_3 = 10$.
Arrangement $(3, 1, 2)$:

- First patient doesn't wait
- Second patient waits for 10 minutes
- Third patient waits for $10 + 15 = 25$ minutes
- Total waiting time $10 + 25 = 35$ minutes

Greedy Strategy

- Make some greedy choice
- Reduce to a smaller problem
- Iterate

Greedy Choice

- First treat the patient with the maximum treatment time
- First treat the patient with the minimum treatment time
- First treat the patient with average treatment time

Greedy Algorithm

- First treat the patient with the minimum treatment time

Greedy Algorithm

- First treat the patient with the minimum treatment time
- Remove this patient from the queue

Greedy Algorithm

- First treat the patient with the minimum treatment time
- Remove this patient from the queue
- Treat all the remaining patients in such order as to minimize their total waiting time

Definition

Subproblem is a similar problem of smaller size.

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$ ``9'' $+$

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$ `‘‘9’’` $+ \texttt{MaximumSalary}(1, 8, 9, 6)$

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$ $\texttt{``9''} + \texttt{MaximumSalary}(1, 8, 9, 6)$
- Minimum total waiting time for *n* patients $=$

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$ ``` ``9'' ``` $+ \texttt{MaximumSalary}(1, 8, 9, 6)$
- Minimum total waiting time for n patients $= (n-1) \cdot t_{min} +$

Subproblem

Examples

- $\texttt{MaximumSalary}(1, 9, 8, 9, 6) =$ $\texttt{``9''} + \texttt{MaximumSalary}(1, 8, 9, 6)$
- Minimum total waiting time for n patients $= (n - 1) \cdot t_{min} +$ minimum total waiting time for $n - 1$ patients without t_{min}

Safe Choice

Definition

A greedy choice is called safe choice if there is an optimal solution consistent with this first choice.

Lemma

To treat the patient with minimum treatment time t_{min} first is a safe choice.

Proof Idea

Is it possible for an optimal arrangement to have two consecutive patients in order with treatment times t_1 and t_2 such that $t_1 > t_2$?

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Is it possible for an optimal arrangement to have two consecutive patients in order with treatment times t_1 and t_2 such that $t_1 > t_2$?

It is impossible. Assume there is such an optimal arrangement and consider what happens if we swap these two patients.

Proof Idea

If we swap two consecutive patients with treatment times $t_1 > t_2$:

- Waiting time for all the patients before and after these two doesn't change

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Proof Idea

If we swap two consecutive patients with treatment times $t_1 > t_2$:

- Waiting time for all the patients before and after these two doesn't change
- Waiting time for the patient which was first increases by t_2, and for the second one it decreases by t_1
- Total waiting time increases by $t_2 - t_1 < 0$, so it actually decreases

Proof Idea

We have just proved:

Lemma

In any optimal arrangement of the patients, first of any two consecutive patients has smaller treatment time.

Safe Choice Proof

- Assume the patient with treatment time t_{min} is not the first

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- Assume the patient with treatment time t_{min} is not the first
- Let $i > 1$ be the position of the first patient with treatment time t_{min} in the optimal arrangement
- Then the patient at position $i - 1$ has bigger treatment time — a contradiction $\square$

Conclusion

Now we know that the following greedy algorithm works correctly:

- First treat the patient with the minimum treatment time

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- Treat all the remaining patients in such order as to minimize their total waiting time

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MinTotalWaitingTime(t, n)

```
waitingTime ← 0
treated ← array of n zeros
for i from 1 to n:
    t_min ← +∞
    minIndex ← 0
    for j from 1 to n:
        if treated[j] == 0 and t[j] < t_min:
            t_min ← t[j]
            minIndex ← j
    waitingTime ← waitingTime + (n − i) · t_min
    treated[minIndex] = 1
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The running time of `MinTotalWaitingTime`(t, n) is $O(n^2)$.

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- For each value of i, j changes from 1 to n

Lemma

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Proof

- i changes from 1 to n
- For each value of i, j changes from 1 to n
- This results in $O(n^2)$ □

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- Instead of choosing the patient with minimum treatment time out of remaining ones n times, sort patients by increasing treatment time
- This sorted arrangement is optimal
- It is possible to sort n patients in time $O(n \log n)$ — you will learn how in the next module

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Reduction to Subproblem

- Make some first choice
- Then solve a problem of the same kind
- Smaller: fewer digits, fewer patients
- This is called a "subproblem"

Safe choice

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Safe choice

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- Not all first choices are safe

Safe choice

- A choice is called safe if there is an optimal solution consistent with this first choice
- Not all first choices are safe
- Greedy choices are often unsafe

General Strategy

Problem

General Strategy

Problem $\xrightarrow{\text{greedy choice}}$

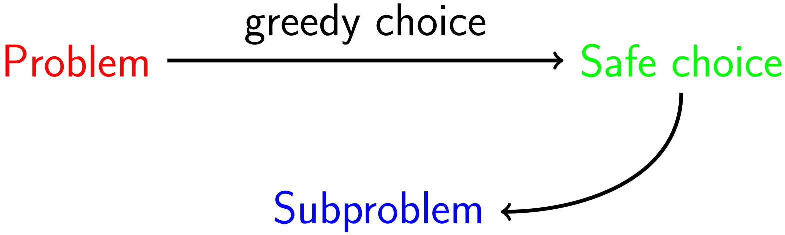
- Make a greedy choice

General Strategy

Problem $\xrightarrow{\text{greedy choice}}$ Safe choice

- Make a greedy choice
- Prove that it is a safe choice

General Strategy



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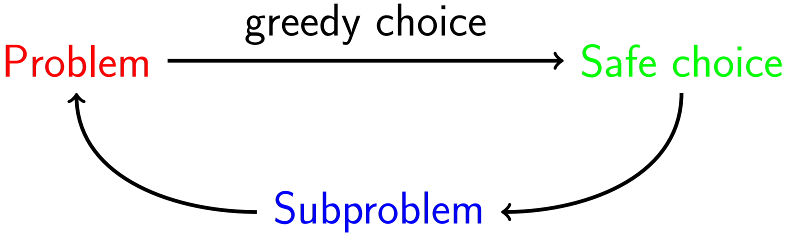
General Strategy



- Make a greedy choice
- Prove that it is a safe choice
- Reduce to a subproblem
- Solve the subproblem