Greedy Algorithms: Celebration Party

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Outline

1. Celebration Party Problem
2. Greedy Algorithm
3. Implementation and Analysis

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differs by at most two years.

Naive Algorithm

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- For each distribution, check whether any two children in any group differ by at most 2 years of age
- Return the minimum number of groups among valid distributions

Running time

Lemma

The running time of the naive algorithm is at least 2^n, where n is the number of children.

Proof

This algorithm will consider all possible distributions of children into two groups (and many other distributions of children into groups). First of these two groups corresponds to any subset of children, and there are 2^n different subsets.

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- We will improve this significantly

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Covering points by segments

Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

Output: The minimum number of segments of length at most 2 needed to cover all the points.

Example

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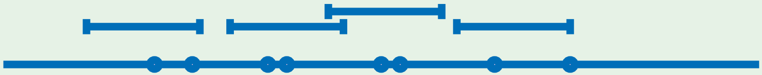

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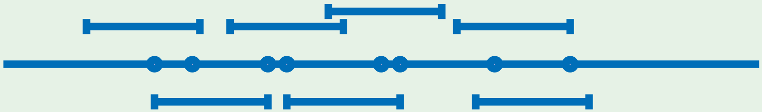

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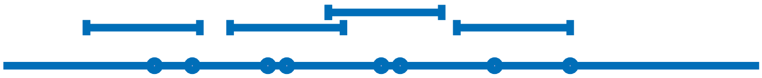
- Points $x_1, \ldots, x_n$ correspond to children' ages
- Segments correspond to groups
- Any two children within the same segment of length 2 differ by at most 2 years of age
- Any valid group of children can be put into a segment of length 2

Safe choice: cover the leftmost point with a segment of length 2 which starts in this point.



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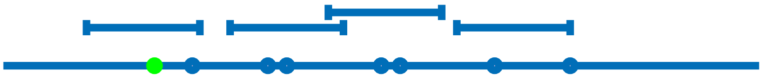

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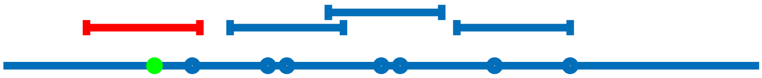

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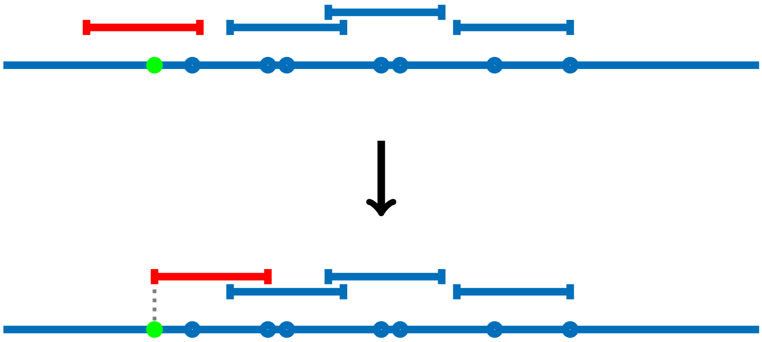

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- Solve the same problem with the remaining points

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Assume $x_1 \leq x_2 \leq \ldots \leq x_n$

PointsCoverSorted$(x_1, \ldots, x_n)$

```
segments ← empty list
left ← 1
while left ≤ n:
    (ℓ, r) ← (x_left, x_left + 2)
    segments.append((ℓ, r))
    left ← left + 1
    while left ≤ n and x_left ≤ r:
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Proof

- *left* changes from 1 to n
- For each *left*, append at most 1 new segment to solution
- Overall, running time is $O(n)$ □

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- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
- Soon you'll learn to sort in $O(n \log n)$
- Sort + `PointsCoverSorted` is $O(n \log n)$

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- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe choice is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$