Greedy Algorithms: Maximizing Loot

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Outline

Maximizing Loot

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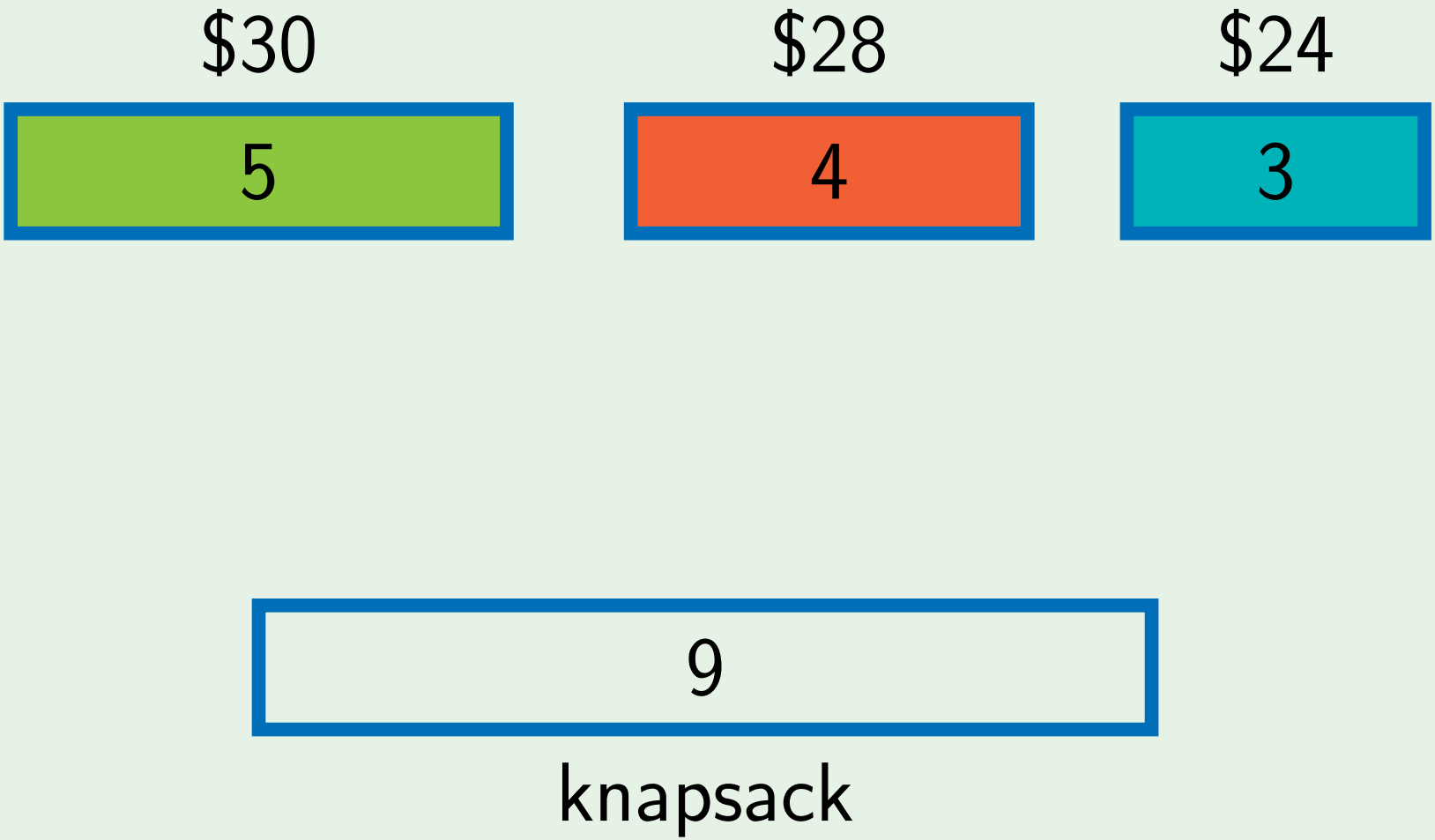
Fractional knapsack

Input: Weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$ of n items; capacity W.

Output: The maximum total value of fractions of items that fit into a knapsack of capacity W.

Example



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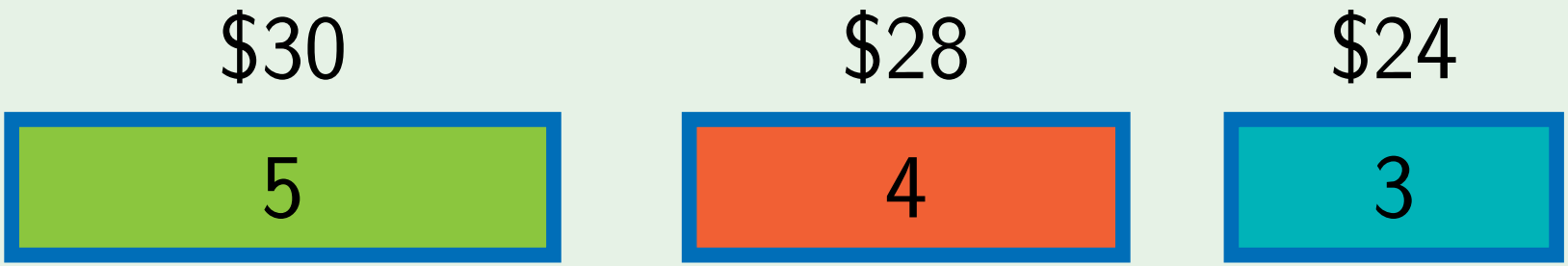

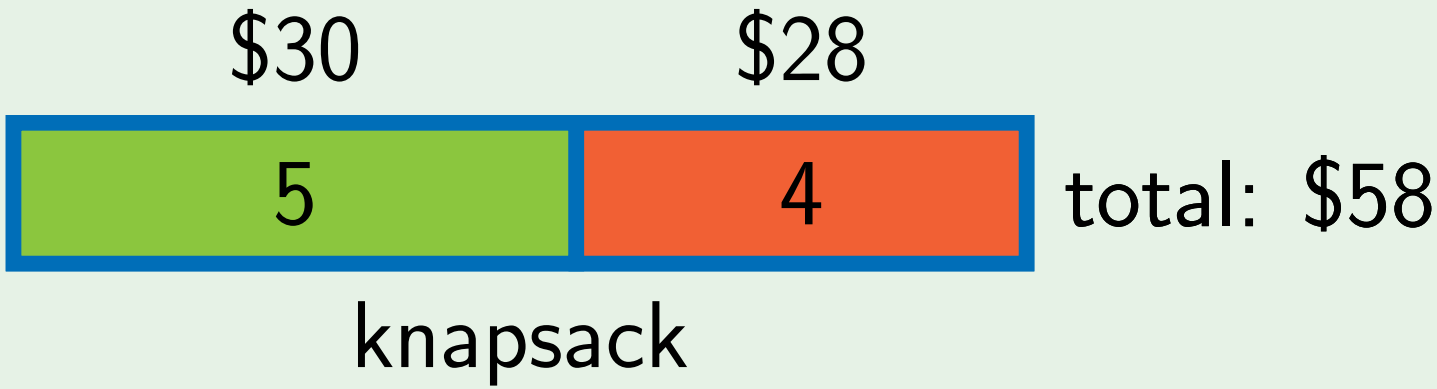

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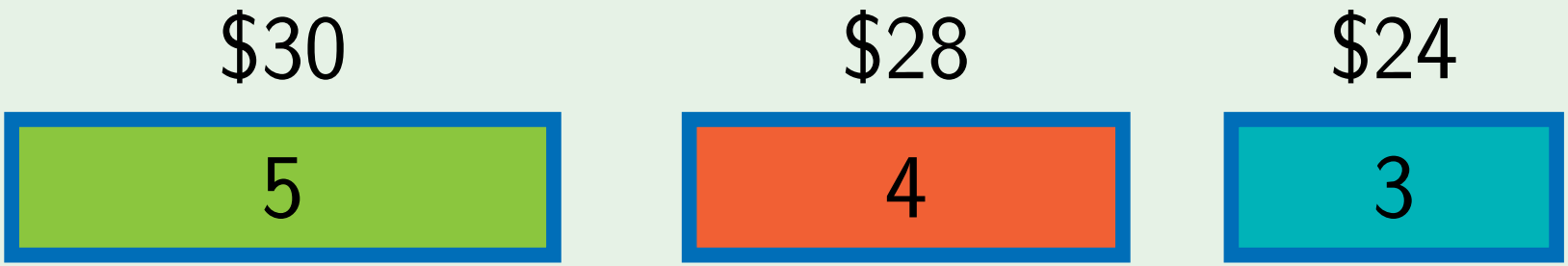

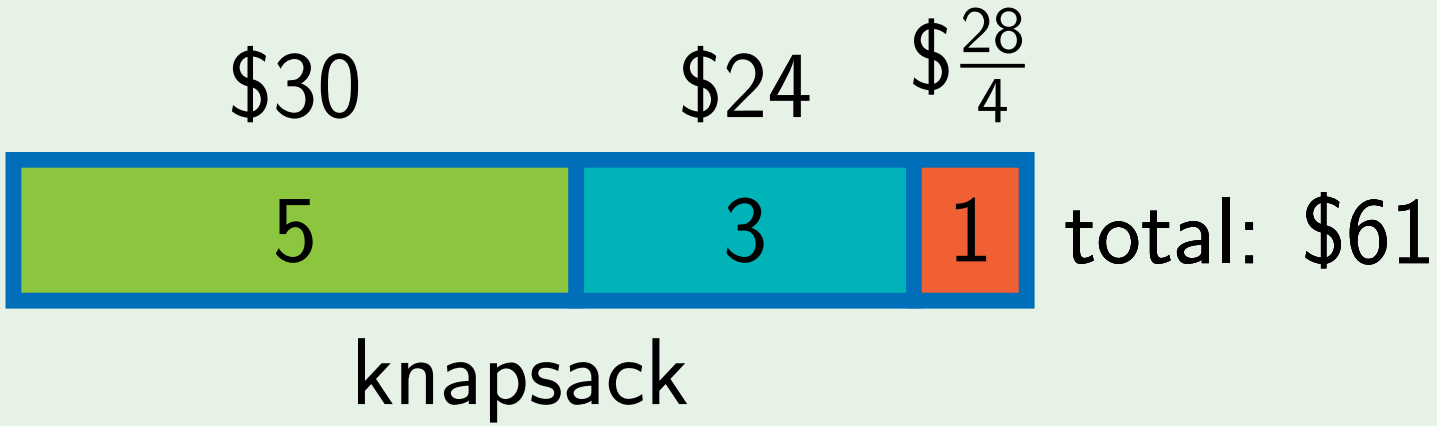

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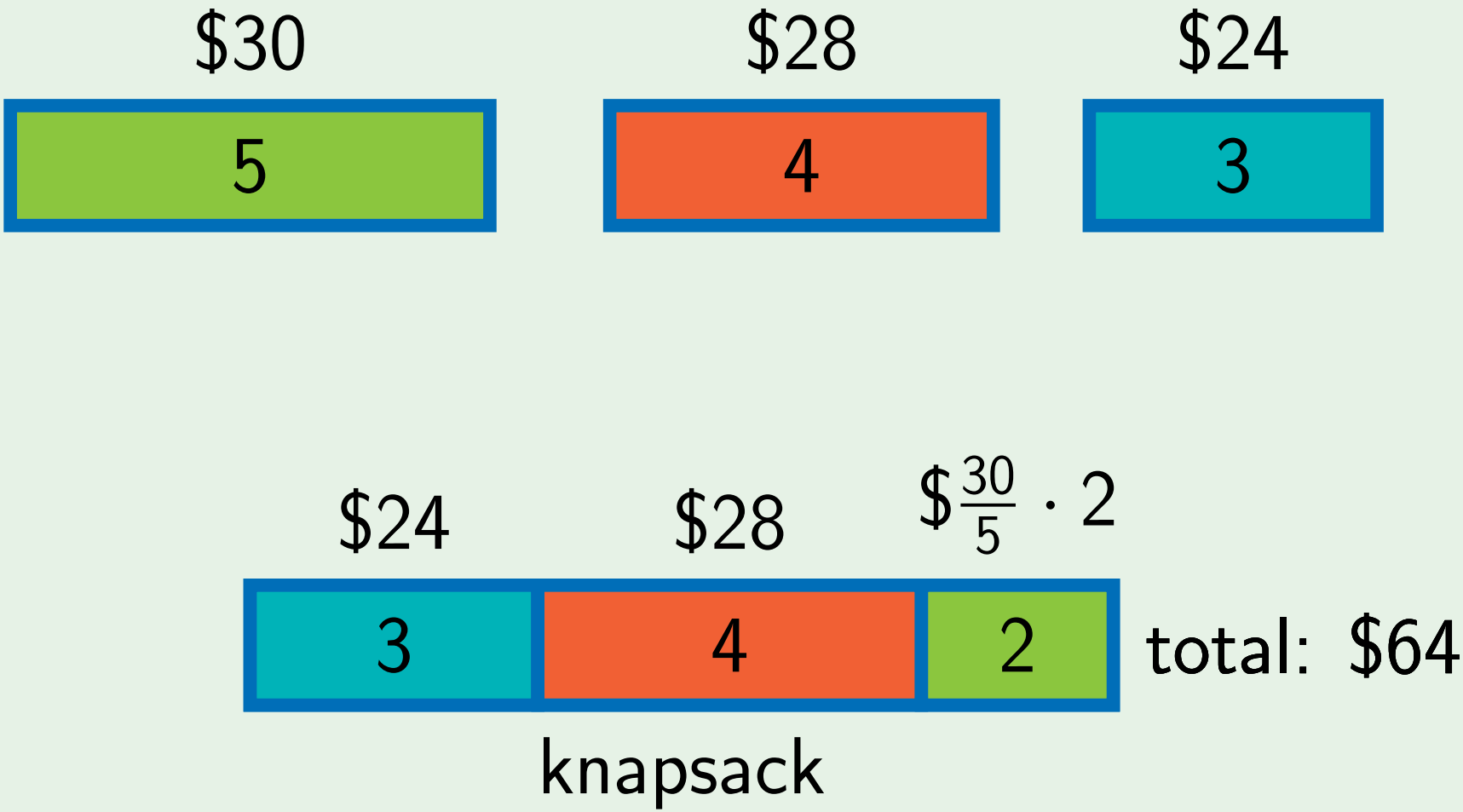

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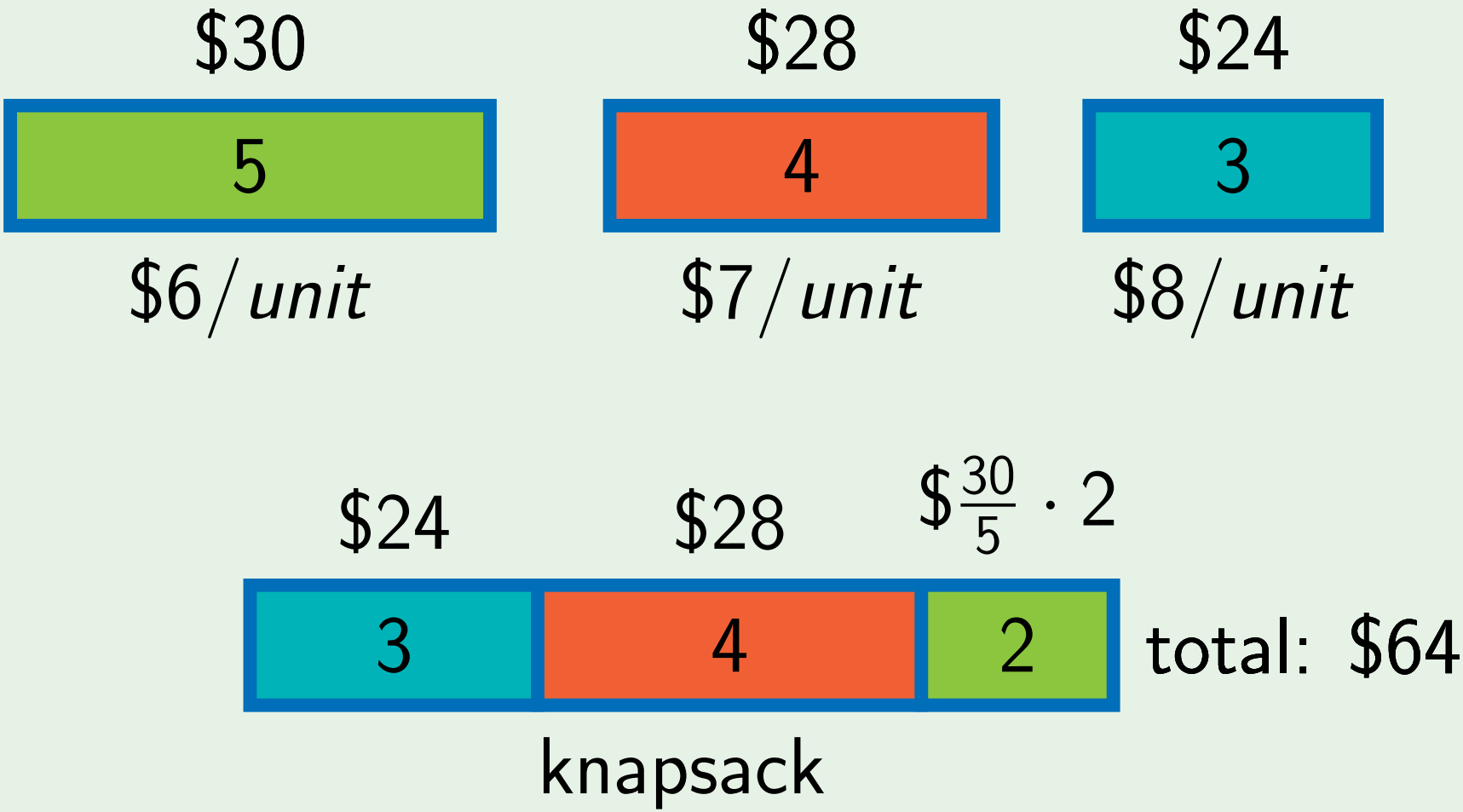
$30
5
$6/unit
$28
4
$7/unit
$24
3
$8/unit
$24
$28
$\frac{30}{5} \cdot 2$
3
4
2
total: $64
knapsack

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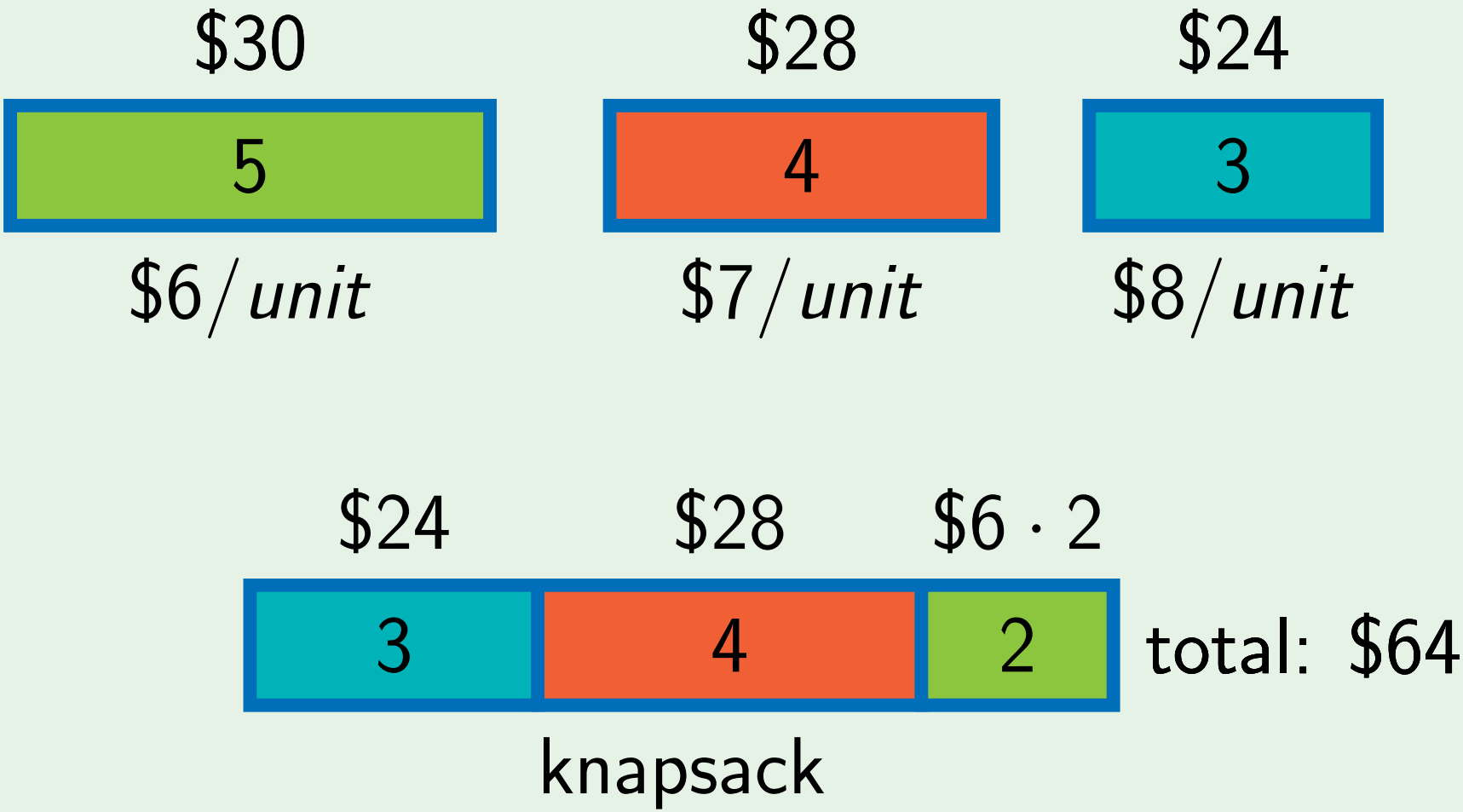
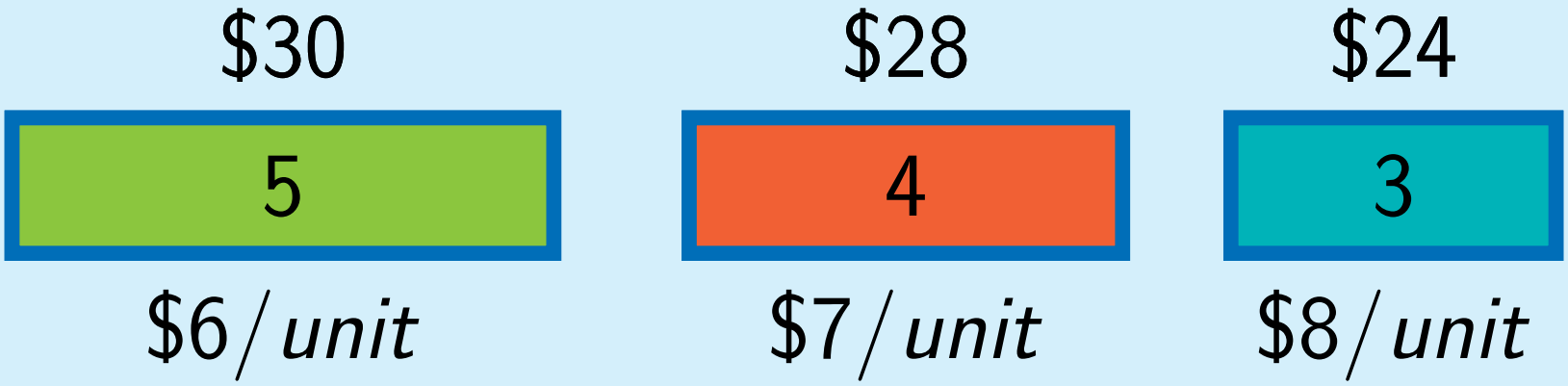


Safe choice

Lemma

There exists an optimal solution that uses as much as possible of an item with the maximal value per unit of weight.

Proof



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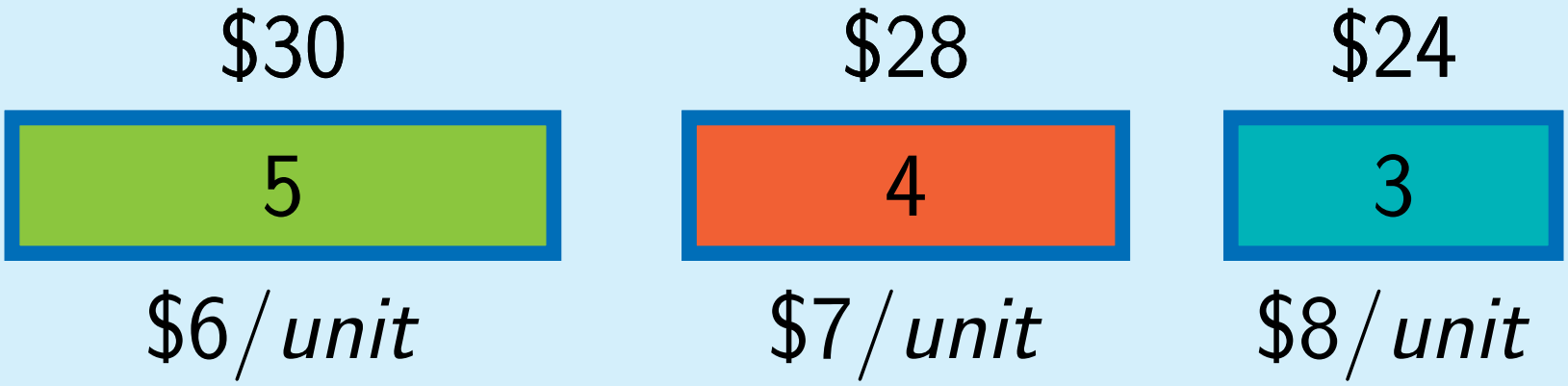

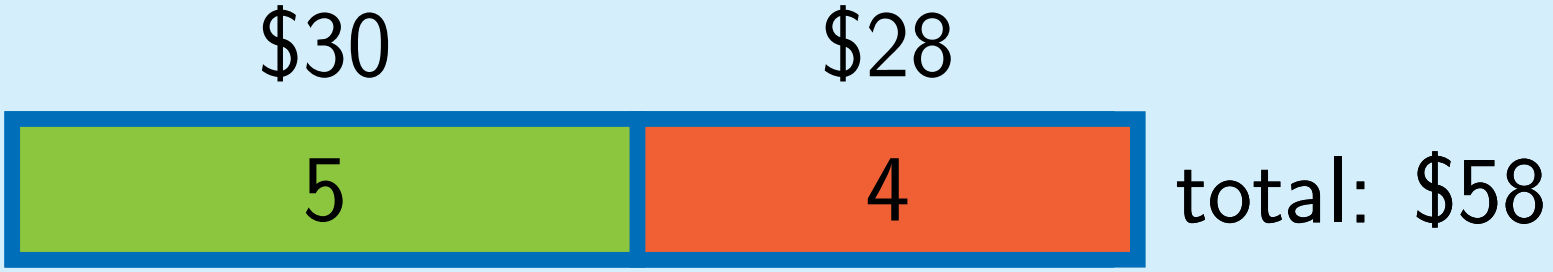

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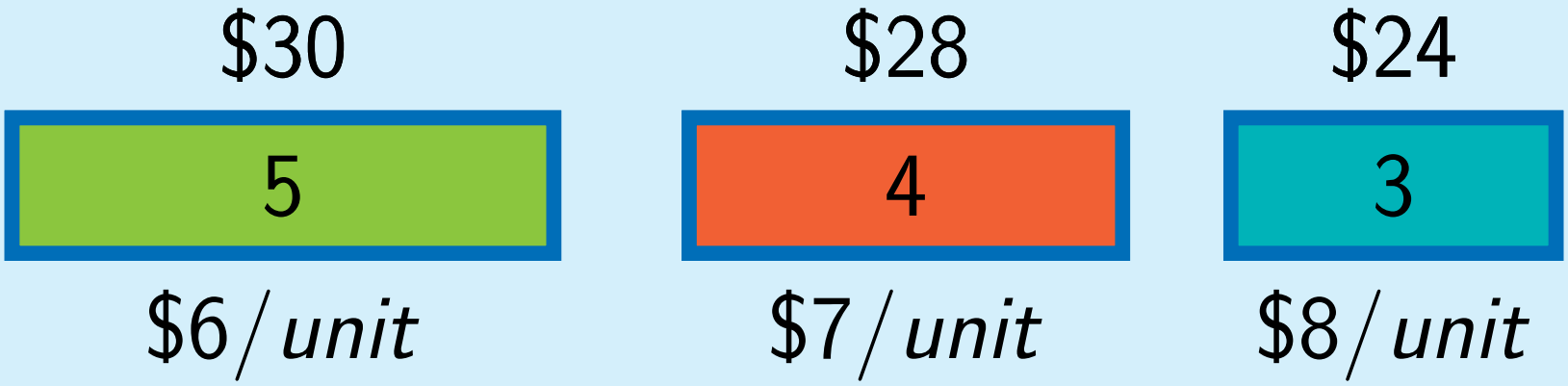

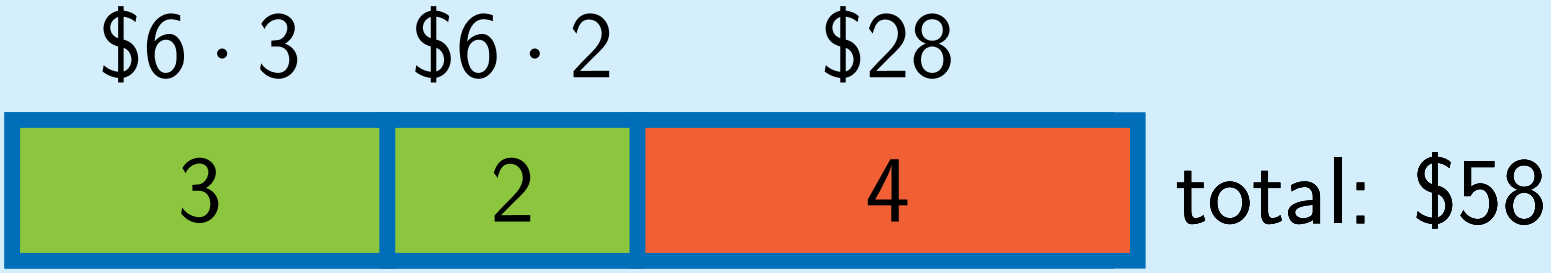

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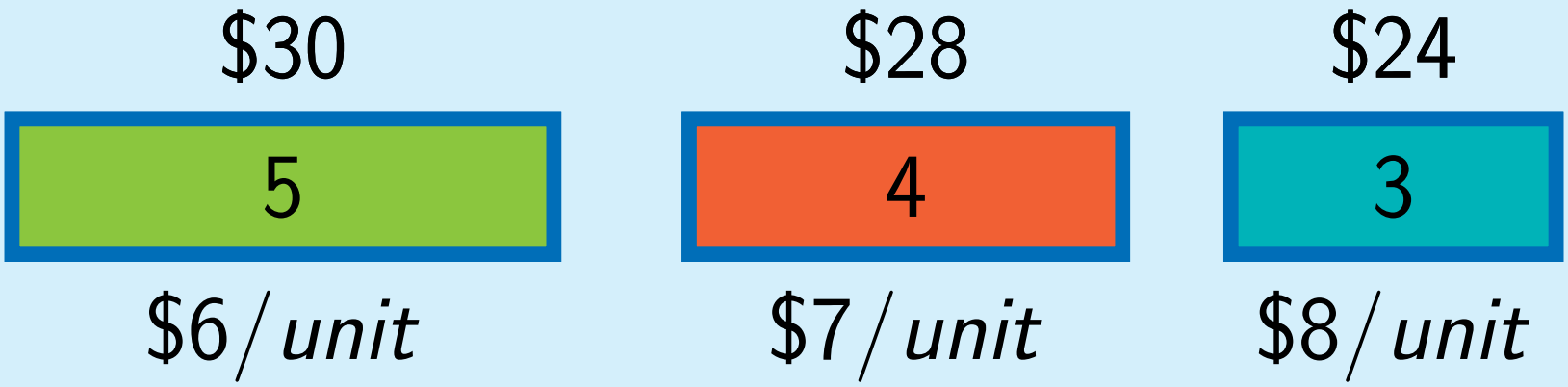
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$28
$24
5
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$6/unit
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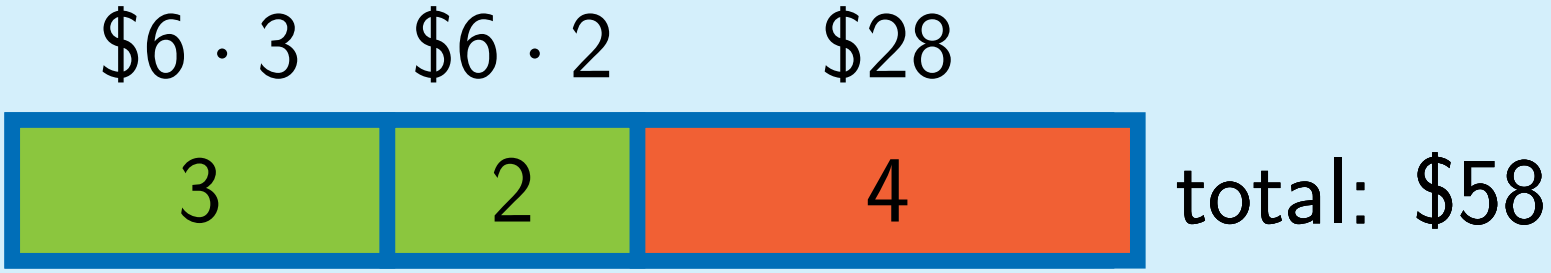
$6 · 3
$6 · 2
$28
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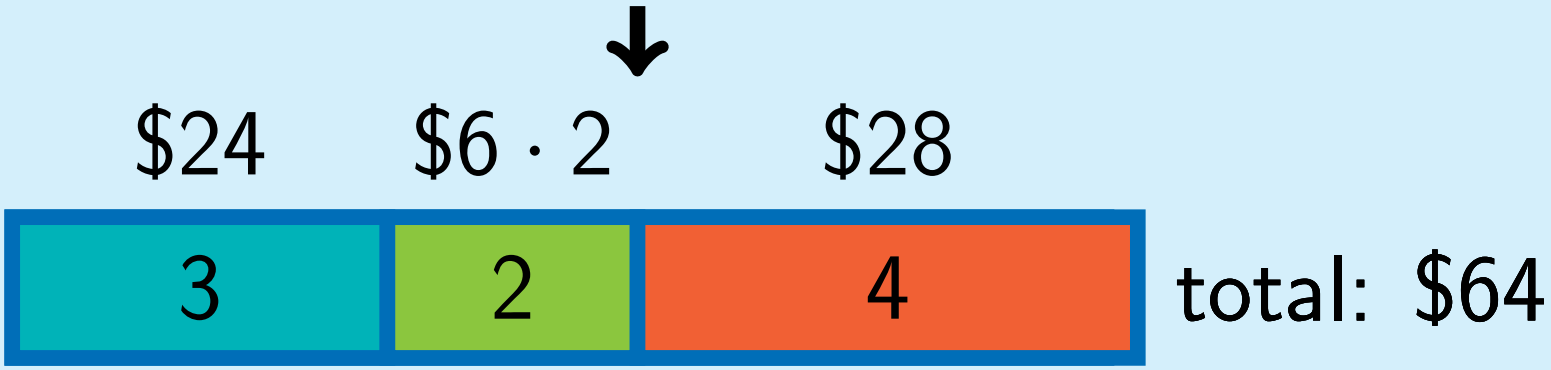
total: $58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BestItem($w_1, v_1, \ldots, w_n, v_n$)

$maxValuePerWeight \leftarrow 0$

$bestItem \leftarrow 0$

for i from 1 to n:

  if $w_i > 0$:

    if $\frac{v_i}{w_i} > maxValuePerWeight$:

      $maxValuePerWeight \leftarrow \frac{v_i}{w_i}$

      $bestItem \leftarrow i$

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for i from 1 to n:
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- Overall, $O(n^2)$ □

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Assume $\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$

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Asymptotics

- Now each iteration is $O(1)$
- `Knapsack` after sorting is $O(n)$
- Sort + `Knapsack` is $O(n \log n)$