Divide-and-Conquer: Searching in an Array

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Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

a problem to be solved

Divide: Break into non-overlapping subproblems of the same type

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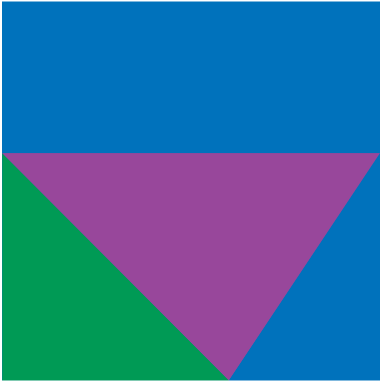
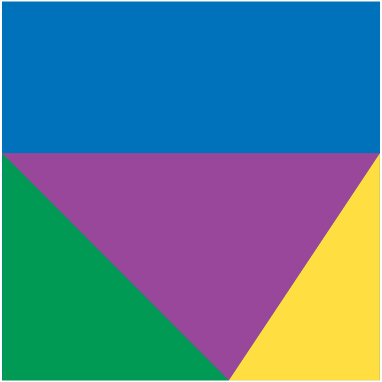
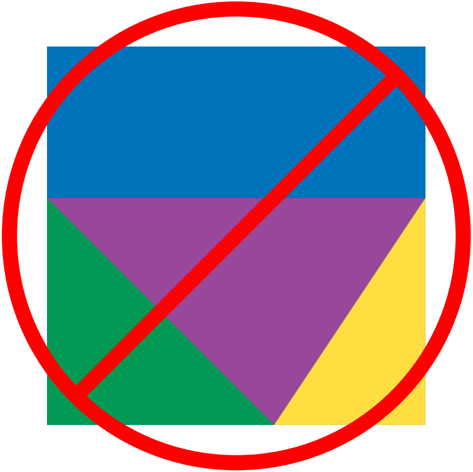
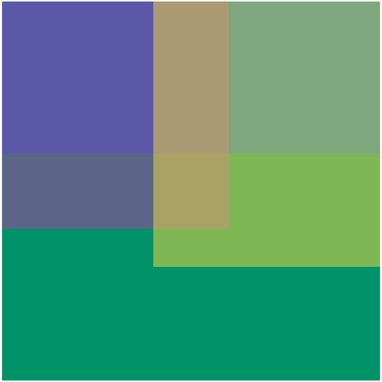
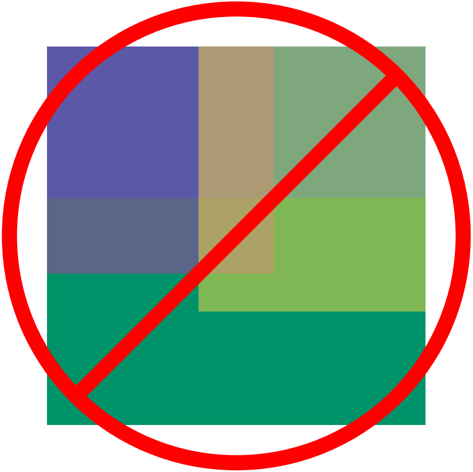






not the
same type





overlapping

Divide: break apart

Divide: break apart

Conquer: solve subproblems

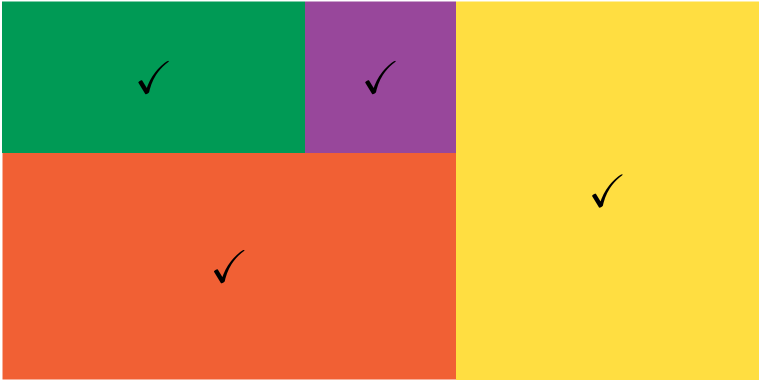
Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: combine



✓

1. Break into non-overlapping subproblems of the same type
2. Solve subproblems
3. Combine results

Outline

Linear Search in Array

Ann	Pat	...	Joe	Bob

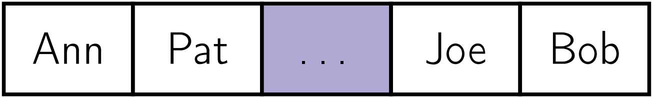
Linear Search in Array

Ann	Pat	. . .	Joe	Bob

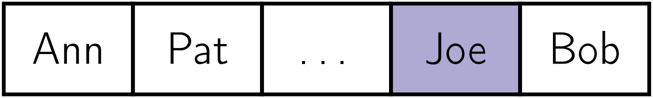
Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array



Linear Search in Array



Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
car	voiture	auto	Auto	auto
table	table	tavola	Tabelle	mesa

Searching in an array

Input: An array A with n elements.
A key k.

Output: An index, i, where $A[i] = k$.
If there is no such i, then NOT_FOUND.

Recursive Solution

$\texttt{LinearSearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

Recursive Solution

`LinearSearch`(A, low, $high$, key)

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
```

Recursive Solution

LinearSearch$(A, low, high, key)$

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if high < low:
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if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recursive Solution

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Definition

A recurrence relation is an equation recursively defining a sequence of values.

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Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

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$$0, 1, 1, 2, 3, 5, 8, \ldots$$

$\texttt{LinearSearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

LinearSearch(A, *low*, *high*, *key*)

```
if high < low:
    return NOT_FOUND
if A[low] = key:
    return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

$$T(n) = T(n-1) + c$$

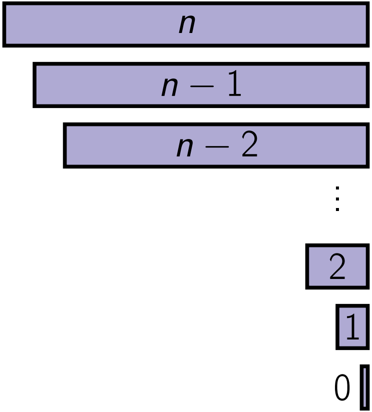
LinearSearch(A, low, $high$, key)

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

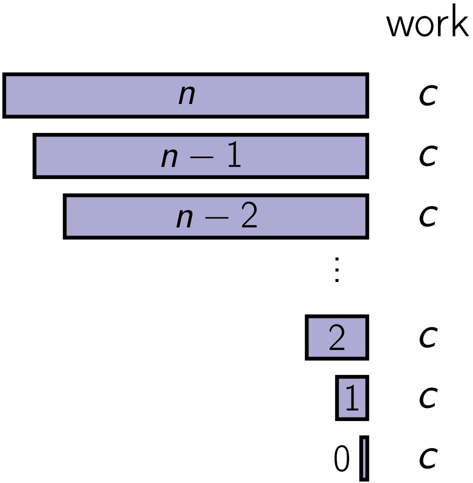
Recurrence defining worst-case time:

$$T(n) = T(n-1) + c$$
$$T(0) = c$$

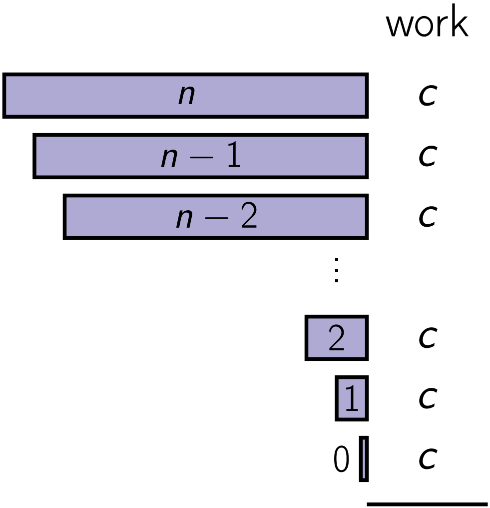
Runtime of Linear Search



Runtime of Linear Search



Runtime of Linear Search



Total: $\sum_{i=0}^{n} c = \Theta(n)$

Iterative Version

LinearSearchIt(A, low, $high$, key)

```
for i from low to high:
  if A[i] = key:
    return i
return NOT_FOUND
```

Summary

- Create a recursive solution

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- Define a corresponding recurrence relation, T

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- Determine $T(n)$: worst-case runtime

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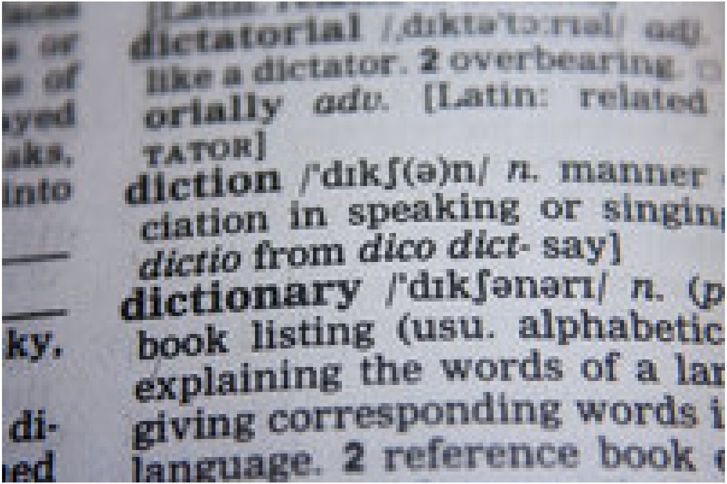
- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime
- Optionally, create iterative solution

Outline

Searching Sorted Data



Searching in a sorted array

Input: A sorted array $A[low \ldots high]$ $(\forall low \leq i < high\colon A[i] \leq A[i+1])$.
A key k.

Output: An index, i, $(low \leq i \leq high)$ where $A[i] = k$.
Otherwise, the greatest index i, where $A[i] < k$.
Otherwise $(k < A[low])$, the result is $low - 1$.

Searching in a Sorted Array

Example

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

search(2) $\rightarrow$ 0

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$

$search(3) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$
$search(3) \to 1$
$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \rightarrow 0$ $search(20) \rightarrow 4$

$search(3) \rightarrow 1$

$search(4) \rightarrow 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$\textit{search}(2) \to 0$ $\quad$ $\textit{search}(20) \to 4$

$\textit{search}(3) \to 1$ $\quad$ $\textit{search}(20) \to 5$

$\textit{search}(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$$\begin{array}{ll} search(2) \to 0 & search(20) \to 4 \\ search(3) \to 1 & search(20) \to 5 \\ search(4) \to 1 & search(60) \to 7 \end{array}$$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$$\begin{array}{ll} \mathit{search}(2) \to 0 & \mathit{search}(20) \to 4 \\ \mathit{search}(3) \to 1 & \mathit{search}(20) \to 5 \\ \mathit{search}(4) \to 1 & \mathit{search}(60) \to 7 \\ & \mathit{search}(70) \to 7 \end{array}$$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

BinarySearch(A, *low*, *high*, *key*)

$\texttt{BinarySearch}(\mathit{A}, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
  return low − 1
```

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

$\texttt{if}\ \mathit{high} < \mathit{low}:$

$\quad \texttt{return}\ \mathit{low} - 1$

$\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$

$\texttt{BinarySearch}(A, low, high, key)$

$\texttt{if}\ \ high < low:$

$\quad \texttt{return}\ \ low - 1$

$mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$

$\texttt{if}\ \ key = A[mid]:$

$\quad \texttt{return}\ \ mid$

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
    return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
```

BinarySearch$(A, low, high, key)$

```
if high < low:
    return low − 1
mid ← ⌊low + (high − low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
else:
    return BinarySearch(A, mid + 1, high, key)
```

Example: Searching for the key 50

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Example: Searching for the key 50

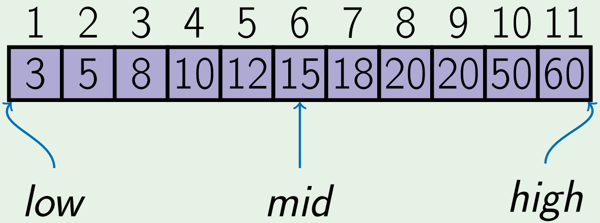
`BinarySearch`$(A, 1, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *high*

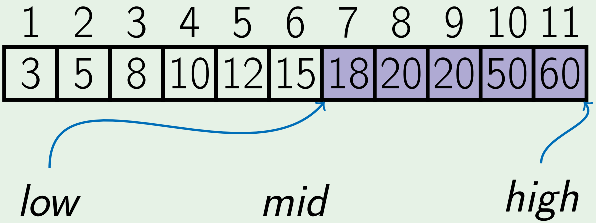
Example: Searching for the key 50

BinarySearch$(A, 1, 11, 50)$



Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$



Example: Searching for the key 50

`BinarySearch`$(A, 1, 11, 50)$
`BinarySearch`$(A, 7, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50) \to 10$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Summary

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- Break problem into non-overlapping subproblems of the same type.

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- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.
- Combine results of subproblems.

BinarySearch($A, low, high, key$)

```
if high < low:
    return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
    return mid
else if key < A[mid]:
    return BinarySearch(A, low, mid − 1, key)
else:
    return BinarySearch(A, mid + 1, high, key)
```

Binary Search Recurrence Relation

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$

Binary Search Recurrence Relation

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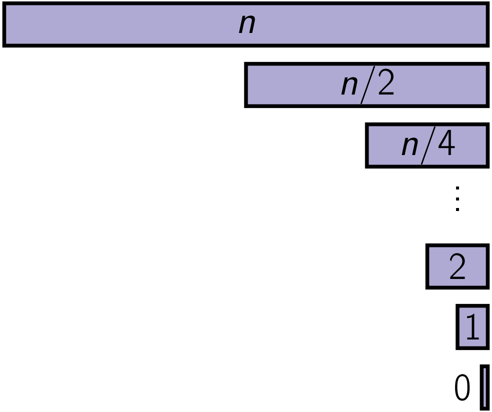
Binary Search Recurrence Relation

$$
\begin{aligned}
T(n) &= T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c \\
T(0) &= c
\end{aligned}
$$

Runtime of Binary Search



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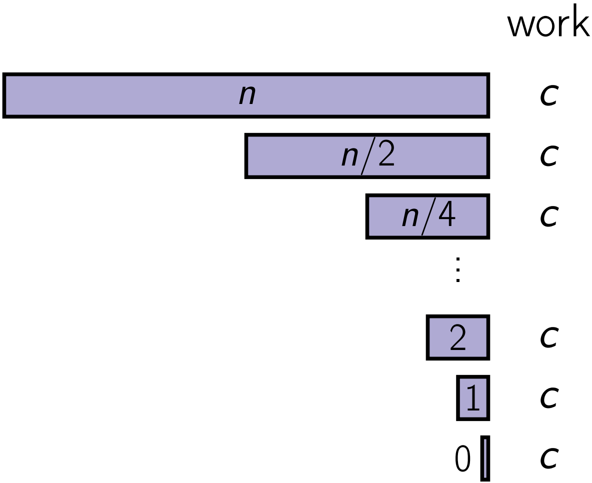
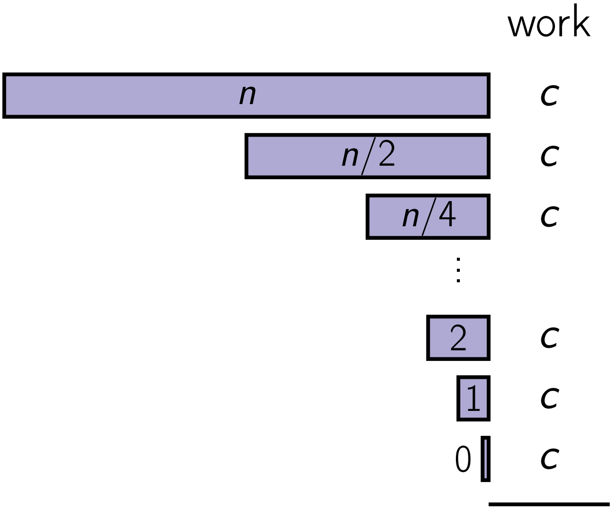


Runtime of Binary Search



$$\text{Total: } \sum_{i=0}^{\log_2 n} c = \Theta(\log_2 n)$$

Iterative Version

`BinarySearchIt`$(A, low, high, key)$

`while` $low \le high$:

$\quad mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$

Iterative Version

BinarySearchIt$(A, low, high, key)$

$\texttt{while } low \leq high:$

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$\quad \texttt{if } key = A[mid]:$

$\quad\quad \texttt{return } mid$

Iterative Version

`BinarySearchIt`$(A, low, high, key)$

`while` $low \leq high$:

  $mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$

  `if` $key = A[mid]$:

    `return` mid

  `else if` $key < A[mid]$:

    $high = mid - 1$

Iterative Version

BinarySearchIt$(A, low, high, key)$

```
while low ≤ high:
   mid ← ⌊low + (high−low)/2⌋
   if key = A[mid]:
      return mid
   else if key < A[mid]:
      high = mid − 1
   else:
      low = mid + 1
```

Iterative Version

$\texttt{BinarySearchIt}(A, \mathit{low}, \mathit{high}, \mathit{key})$

$\texttt{while}\ \mathit{low} \le \mathit{high}:$

$\quad \mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high}-\mathit{low}}{2} \right\rfloor$

$\quad \texttt{if}\ \mathit{key} = A[\mathit{mid}]:$

$\quad\quad \texttt{return}\ \mathit{mid}$

$\quad \texttt{else if}\ \mathit{key} < A[\mathit{mid}]:$

$\quad\quad \mathit{high} = \mathit{mid} - 1$

$\quad \texttt{else:}$

$\quad\quad \mathit{low} = \mathit{mid} + 1$

$\texttt{return}\ \mathit{low} - 1$

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

Real-life Example

english (sorted)	**french** (sorted)	**italian** (sorted)	**german** (sorted)	**spanish** (sorted)
chair	chaise	casa	Haus	casa
house	bouton	foruncolo	Pickel	espenilla
pimple	maison	sedia	Sessel	silla

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
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english sorted
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Real-life Example

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english
sorted

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spanish
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sorted

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Real-life Example

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english sorted
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1
3

spanish sorted
1
3
2

Summary

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The runtime of binary search is $\Theta(\log n)$.