Divide-and-Conquer: Quick Sort

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Algorithmic Design and Techniques
Algorithms and Data Structures at edX

Outline

1. Overview
2. Algorithm
3. Random Pivot
4. Running Time Analysis
5. Equal Elements
6. Final Remarks

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
- efficient in practice

Example: quick sort

6	4	8	2	9	3	9	4	7	6	1

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partition with respect to $x = A[1]$

in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9

≤ 6 $\qquad\qquad$ > 6

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in particular, x is in its final position

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sort the two parts recursively

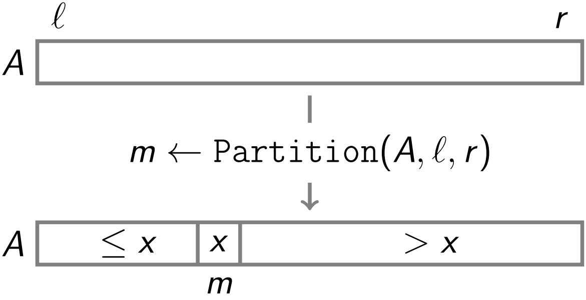
1	2	3	4	4	6	6	7	8	9	9

Outline

```
QuickSort(A, ℓ, r)

if ℓ ≥ r:
  return
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
QuickSort(A, ℓ, m − 1)
QuickSort(A, m + 1, r)
```


ℓ
r
A



ℓ
r
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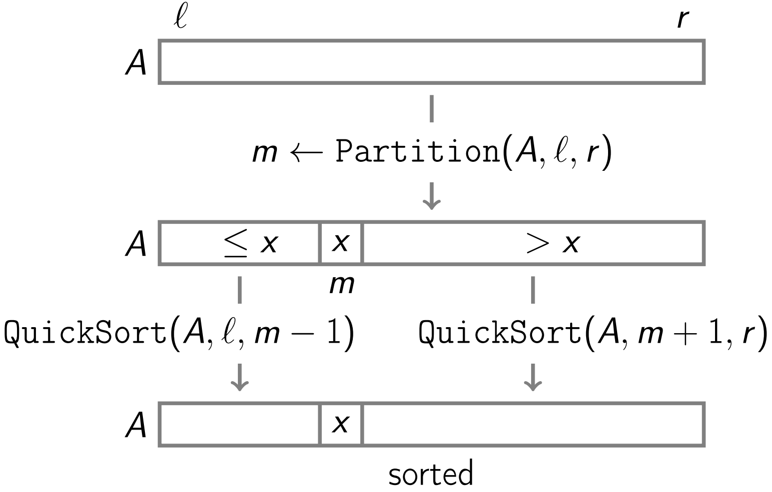
$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
$\leq x$
x
$> x$
m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$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
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x
$> x$
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$\texttt{QuickSort}(A, \ell, m-1)$
$\texttt{QuickSort}(A, m+1, r)$
A
x
sorted

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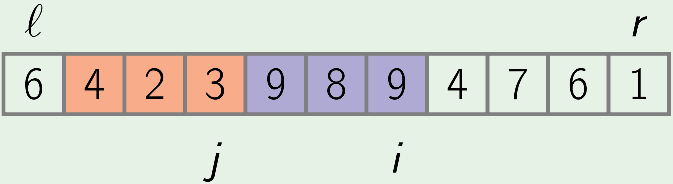

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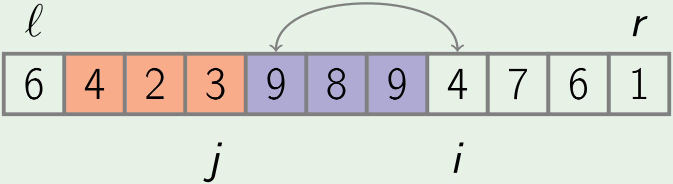

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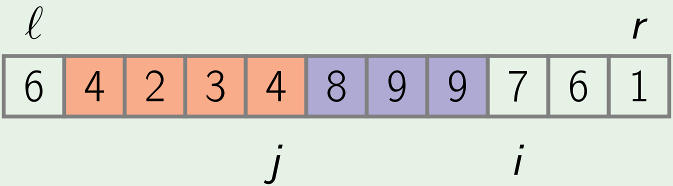

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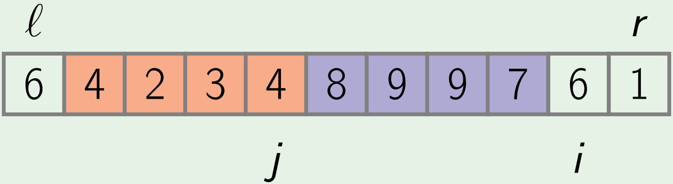

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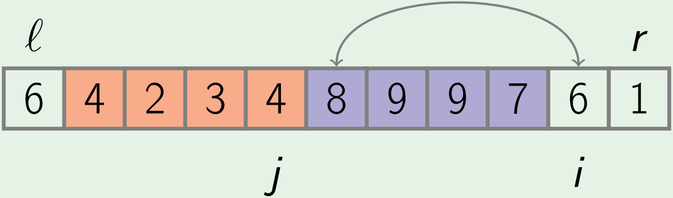

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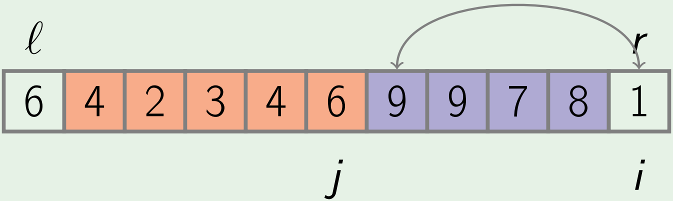

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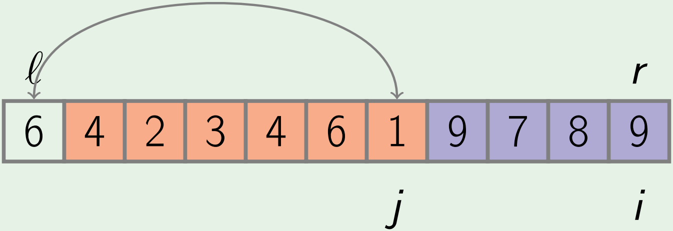

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						j				i

$\texttt{Partition}(A, \ell, r)$

```
x ← A[ℓ]    {pivot}
j ← ℓ
for i from ℓ + 1 to r:
   if A[i] ≤ x:
      j ← j + 1
      swap A[j] and A[i]
   {A[ℓ + 1 . . . j] ≤ x, A[j + 1 . . . i] > x}
swap A[ℓ] and A[j]
return j
```

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- $T(n) = n + T(n-5) + T(4)$:

  $T(n) \geq n+(n-5)+(n-10)+\cdots = \Theta(n^2)$

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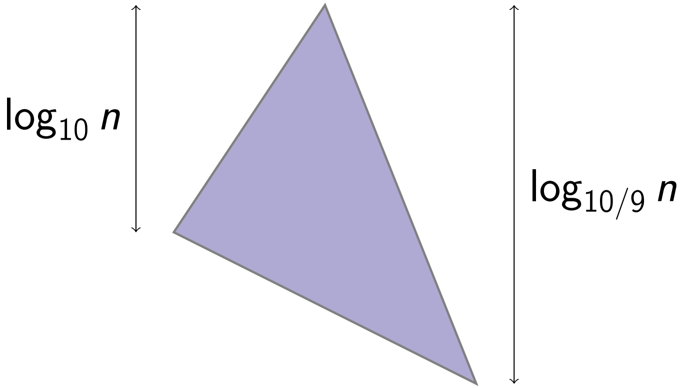

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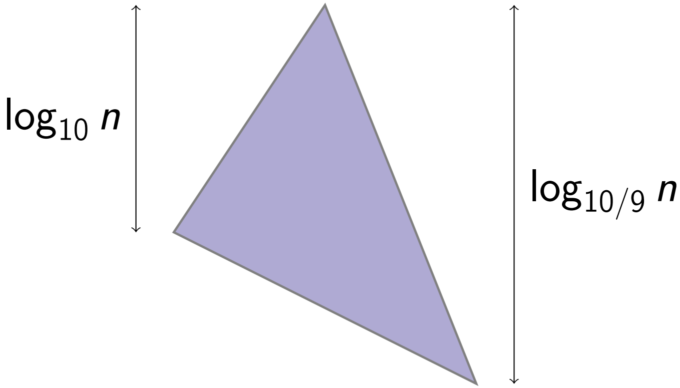
$$T(n) = T(n/10) + T(9n/10) + O(n)$$



$$T(n) = O(n \log n)$$

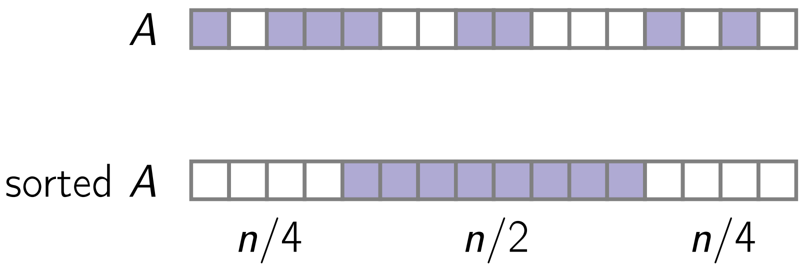
Random Pivot

RandomizedQuickSort(A, ℓ, r)

if $\ell \geq r$:
  return
$k \leftarrow$ random number between ℓ and r
swap $A[\ell]$ and $A[k]$
$m \leftarrow$ Partition(A, ℓ, r)
$\{A[m]$ is in the final position$\}$
RandomizedQuickSort$(A, \ell, m-1)$
RandomizedQuickSort$(A, m+1, r)$

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \dots n]$ are pairwise different. Then the average running time of `RandomizedQuickSort`(A) is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

Outline

Proof Ideas: Comparisons

- the running time is proportional to the number of comparisons made

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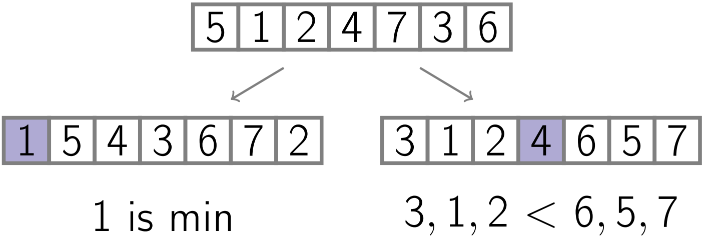
- balanced partition are better since they reduce the number of comparisons needed:



Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

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$$\text{Prob}\,(3 \text{ and } 4 \text{ are compared}) =$$

Proof Ideas: Probability

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A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\left(1 \text{ and } 9 \text{ are compared}\right) = \frac{2}{9}$$

$$\text{Prob}\left(3 \text{ and } 4 \text{ are compared}\right) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$

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- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$
- then $\mathsf{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and $\mathsf{E}(\chi_{ij}) = \frac{2}{j-i+1}$

Proof (continued)

Then (the expected value of) the running time is

$$
\begin{aligned}
\mathsf{E} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \chi_{ij} &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathsf{E}(\chi_{ij}) \\
&= \sum_{i<j} \frac{2}{j-i+1} \\
&\leq 2n \cdot \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) \\
&= \Theta(n \log n)
\end{aligned}
$$

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- what if all the elements of the given array are equal to each other?

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- quick sort visualization
- the array is always split into two parts of size 0 and $n-1$
- $T(n) = n + T(n-1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

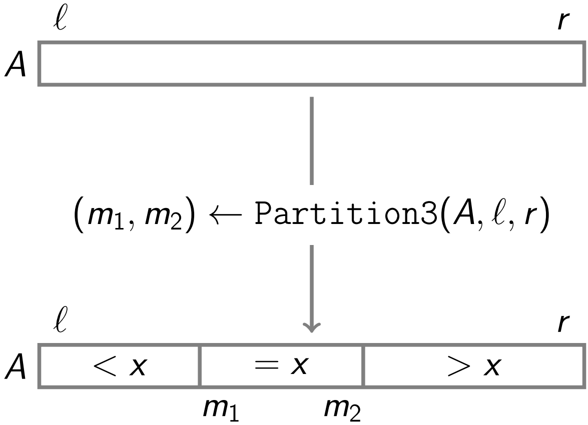
$$m \leftarrow \texttt{Partition}(A, \ell, r)$$

with the line

$$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$



ℓ
r
A
(m1, m2) ← Partition3(A, ℓ, r)
ℓ
r
A
< x
= x
> x
m1
m2

RandomizedQuickSort(A, ℓ, r)

```
if ℓ ≥ r:
  return
k ← random number between ℓ and r
swap A[ℓ] and A[k]
(m₁, m₂) ← Partition3(A, ℓ, r)
{A[m₁ ... m₂] is in final position}
RandomizedQuickSort(A, ℓ, m₁ − 1)
RandomizedQuickSort(A, m₂ + 1, r)
```

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Tail Recursion Elimination

$\texttt{QuickSort}(A, \ell, r)$

```
while ℓ < r:
   m ← Partition(A, ℓ, r)
   QuickSort(A, ℓ, m − 1)
   ℓ ← m + 1
```

QuickSort(A, ℓ, r)

```
while ℓ < r:
   m ← Partition(A, ℓ, r)
   if (m − ℓ) < (r − m):
      QuickSort(A, ℓ, m − 1)
      ℓ ← m + 1
   else:
      QuickSort(A, m + 1, r)
      r ← m − 1
```

$\texttt{QuickSort}(A, \ell, r)$

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Worst-case space requirement: $O(\log n)$

Intro Sort

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- runs quick sort with a simple deterministic pivot selection heuristic (say, median of the first, middle, and last element)
- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

Conclusion

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- Efficient in practice