Dynamic Programming: Change Problem

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Algorithmic Design and Techniques
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Outline

1. Greedy Change
2. Recursive Change
3. Dynamic Programming

Change problem

Find the minimum number of coins needed to make change.

Formally

Change problem

Input: An integer *money* and positive integers $coin_1, \ldots, coin_d$.

Output: The minimum number of coins with denominations $coin_1, \ldots, coin_d$ that changes *money*.

Greedy Way

GreedyChange(*money*)

```
Change ← empty collection of coins
while money > 0:
   coin ← largest denomination
          that does not exceed money
   add coin to Change
   money ← money − coin
return Change
```

Changing Money

in the US

40 cents $=$ $25 + 10 + 5$

Greedy

Changing Money

in Tanzania

$$40 \text{ cents} = 25 + 10 + 5 \quad = \quad 20 + 20$$

Greedy is not Optimal

Outline

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Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

$$MinNumCoins(9) = ?$$

money	1	2	3	4	5	6	7	8	9	10
MinNumCoins									?	

Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

$$MinNumCoins(9) = \min \begin{cases} MinNumCoins(9-6)+1 \\ MinNumCoins(9-5)+1 \\ MinNumCoins(9-1)+1 \end{cases}$$

money	1	2	3	4	5	6	7	8	9	10
MinNumCoins									?	

Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

$$MinNumCoins(9) = \min \begin{cases} MinNumCoins(3) + 1 \\ MinNumCoins(4) + 1 \\ MinNumCoins(8) + 1 \end{cases}$$

money	1	2	3	4	5	6	7	8	9	10
MinNumCoins									?	

Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

$$MinNumCoins(9) = \min \begin{cases} MinNumCoins(3) + 1 \\ MinNumCoins(4) + 1 \\ MinNumCoins(8) + 1 \end{cases}$$

money	1	2	3	4	5	6	7	8	9	10
MinNumCoins			?	?				?	?	

Recurrence for Change Problem

$$MinNumCoins(money) =$$

$$\min \begin{cases} MinNumCoins(money - coin_1) + 1 \\ MinNumCoins(money - coin_2) + 1 \\ \ldots \\ MinNumCoins(money - coin_d) + 1 \end{cases}$$

RecursiveChange($money$, $coins$)

```
if money = 0:
  return 0
MinNumCoins ← ∞
for i from 1 to |coins|:
  if money ≥ coin_i:
    NumCoins ← RecursiveChange(money − coin_i, coins)
    if NumCoins + 1 < MinNumCoins:
      MinNumCoins ← NumCoins + 1
return MinNumCoins
```

How Fast is `RecursiveChange`?

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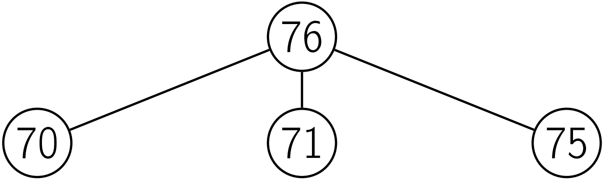
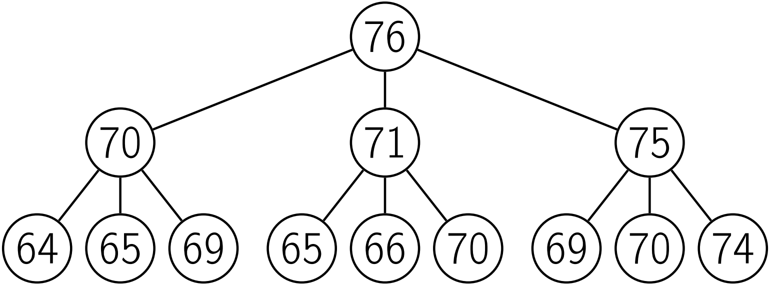
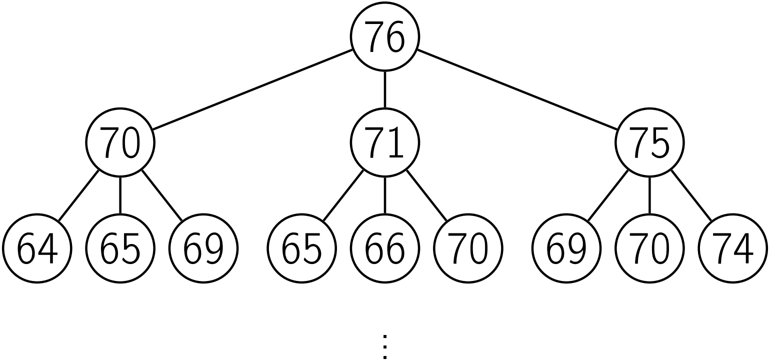
How Fast is `RecursiveChange`?



How Fast is `RecursiveChange`?



How Fast is `RecursiveChange`?



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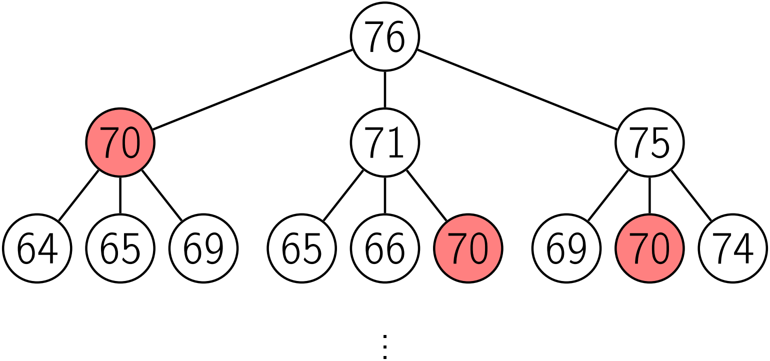
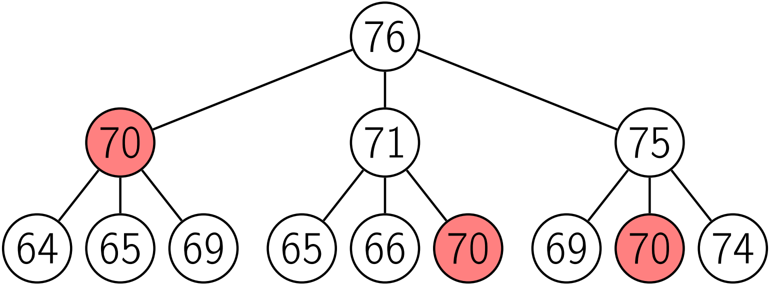
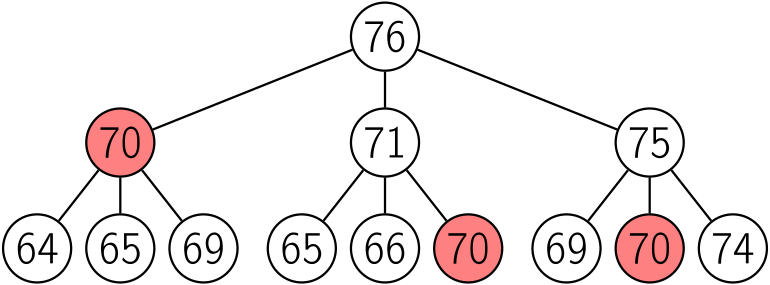


How Fast is `RecursiveChange`?



the optimal coin combination for 70 cents is computed at least **three** times!

How Fast is `RecursiveChange`?



the optimal coin combination for 70 cents is computed at least **three** times!

the optimal coin combination for 30 cents is computed **trillions** of times!

Hint

Wouldn't it be nice to know all the answers for changing $money - coin_i$ by the time we need to compute an optimal way of changing $money$?

Hint

Wouldn't it be nice to know all the answers for changing $money - coin_i$ by the time we need to compute an optimal way of changing $money$?

Instead of the time-consuming calls to

$\texttt{RecursiveChange}(money - coin_i, coins)$

we would simply look up these values!

Outline

1. Greedy Change
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Dynamic Programming

What is the minimum number of coins needed to change 0 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0									

Dynamic Programming

What is the minimum number of coins needed to change 1 cent for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1								

Dynamic Programming

What is the minimum number of coins needed to change 2 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2							

Dynamic Programming

What is the minimum number of coins needed to change 3 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3						

Dynamic Programming

What is the minimum number of coins needed to change 4 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4					

Dynamic Programming

What is the minimum number of coins needed to change 5 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4					

$$\min \begin{cases} \mathit{MinNumCoins}(0) + 1 \\ \mathit{MinNumCoins}(4) + 1 \end{cases}$$

Dynamic Programming

What is the minimum number of coins needed to change 5 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1				

Dynamic Programming

What is the minimum number of coins needed to change 6 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1				

$$\min \begin{cases} MinNumCoins(0) + 1 \\ MinNumCoins(1) + 1 \\ MinNumCoins(5) + 1 \end{cases}$$

Dynamic Programming

What is the minimum number of coins needed to change 6 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1			

Dynamic Programming

What is the minimum number of coins needed to change 7 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1			

Dynamic Programming

What is the minimum number of coins needed to change 7 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1	2		

Dynamic Programming

What is the minimum number of coins needed to change 8 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1	2		

Dynamic Programming

What is the minimum number of coins needed to change 8 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1	2	3	

Dynamic Programming

What is the minimum number of coins needed to change 9 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1	2	3	

Dynamic Programming

What is the minimum number of coins needed to change 9 cents for denominations 6, 5, and 1?

money	0	1	2	3	4	5	6	7	8	9
MinNumCoins	0	1	2	3	4	1	1	2	3	4

DPChange($money$, $coins$)

```
MinNumCoins(0) ← 0
for m from 1 to money:
  MinNumCoins(m) ← ∞
  for i from 1 to |coins|:
    if m ≥ coin_i:
      NumCoins ← MinNumCoins(m − coin_i) + 1
      if NumCoins < MinNumCoins(m):
        MinNumCoins(m) ← NumCoins
return MinNumCoins(money)
```

"Programming" in "Dynamic Programming" Has Nothing to Do with Programming!

Richard Bellman developed this idea in 1950s working on an Air Force project.

At that time, his approach seemed completely impractical.

He wanted to hide that he is really doing math from the Secretary of Defense.

Richard Bellman

...What name could I choose? I was interested in planning but planning, is not a good word for various reasons. I decided therefore to use the word, "programming" and I wanted to get across the idea that this was dynamic. **It was something not even a Congressman could object to.** So I used it as an umbrella for my activities.